

Qubits, Weyl spinors, quantum NOT gates, and dynamical decoupling

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An equivalence is established between orthogonal pure state qubits on the Bloch sphere and massless Weyl spinors, when the Bloch vector is taken as the physical three-momentum. A family of unitary, coordinate dependent transformations is obtained which connects orthogonal combinations of the basis states of a two-level quantum system. It is shown that a subset of these transformations possesses the novel feature of effecting a point inversion by means of a rotation. For qubits, these transformations act as quantum NOT/parity gates, and also as flipping operators that exactly cancel decoherence in a dynamical decoupling setting. For Weyl spinors they provide, at the relativistic quantum level, a unitary symmetry transformation for the Weyl equations.

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I. INTRODUCTION

In quantum information theory the analogue of a classical bit, a qubit, is provided by a general linear complex combination of the orthogonal states of a two-level system. The orthogonal states $|0\rangle$ and $|1\rangle$ of the two-level system constitute the computational basis. If the computational basis is assigned to the north and south poles of a unit sphere, known as the Bloch sphere (figure 1), any point on the surface of the sphere represents a pure state qubit[1, 2], with the corresponding orthogonal state given by the point diametrically opposed to it: the antipodal point[3].

A quantum gate is a unitary matrix acting on one or more qubits, and it is the basic building block for quantum circuits. Classically, the negation or NOT gate swaps the logic binaries 0 and 1. In quantum computing, a quantum NOT gate would, ideally, transform an arbitrary qubit, including the computational basis, into its orthogonal state, mimicking the action of its classical counterpart, and thus providing a universal quantum NOT gate, where in this context universality is to be understood as the ability to output the orthogonal state to any given input[4]. However, it is known that no *fixed* unitary matrix exists for that purpose[2, 3, 5, 6].

A quantum NOT gate is analogous to a spin flip operator[7], and because of the antipodal character of pure state qubits, also to a parity operator. In fact, the universal quantum NOT gate, as described above, can be characterized as a point inversion through the origin of the Bloch sphere[5]. Flipping operators find important applications in dynamical decoupling schemes, where they are used to reverse decoherence along a given axis of the Bloch sphere. This is done by applying sequences of pulses that, on average, transform the state to its mirror state across the relevant symmetry plane[8]. The pulses must anti-commute with the effective interaction Hamiltonian of system plus bath, and thus an operator that anti-commutes with a general interaction Hamiltonian would then constitute a universal dynamical decoupler.

On a seemingly unrelated topic, it is well known that the free Dirac equation decouples in the massless limit into the Weyl equations[9–11], whose solutions are two-component spinors of definite helicity, which is the spin projection along the direction of the physical momentum: left-handed spinors for helicity -1 , and right-handed spinors for helicity $+1$. These classical, c -number spinors, are also regarded as twistors[12]. Left and right-handed spinors belong to nonequivalent irreducible representations of the Lorentz group[13–15], connected by parity. They are also related by an anti-unitary transformation, the so called Wigner transformation or spin flip operation[15–17]. The Wigner transformation is a symmetry transformation of the Weyl equations, analogous to parity[18–20]. As in the qubit case, there is no fixed unitary matrix that connects Weyl spinors of different helicity.

The purpose of this letter is two-fold: one is to present an analogy between pure state qubits and massless Weyl spinors, thus providing a connection between previously thought unrelated topics, and the second is to present a family of unitary transformations connecting the orthogonal combinations of the basis states of a two-level system, and to show their applications to qubits and Weyl spinors. These transformations depend on the states and hence are continuous, coordinate dependent transformations, and in one instance they have the remarkable and novel property of effecting a point inversion by means of a rotation. For qubits, the transformations provide a family of NOT/parity gates which anti-commute with a general interaction Hamiltonian of system plus bath, and thus exactly cancel decoherence in a dynamical decoupling setting. Also, a one parameter transformation is obtained that satisfy the criterion for universality described above. For Weyl spinors, the transformations comprise a novel symmetry transformation for the Weyl equations.

The organization is as follows: section II provides the relation between qubits and massless Weyl spinors, and it is also pointed out that there are two types of spinors with different transformation phases under a discrete parity transformation \mathcal{P} . Section III introduces the unitary transformations and their properties in a general setting. Section IV deals with the application of the transformations to qubits, and shows their role in dynamical decoupling. Section V presents the new symmetry transformation of the Weyl equations, and also shows the role of the transformations in the definition of twistors. Finally, concluding remarks are given. Natural units with $\hbar = c = 1$ are used throughout.

II. QUBITS AS MASSLESS WEYL SPINORS

In regards to the unit Bloch sphere [1, 2], a general qubit is given by

$$|\chi_+\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \quad (1)$$

where θ and φ are the spherical polar angles and $\{|0\rangle, |1\rangle\}$ is a suitable basis of the two-level system, respectively represented as the north and south poles of the sphere (Figure 1). The subscript in $|\chi_+\rangle$ denotes the helicity, as will be explained shortly. For a spin $1/2$ system, and using the Pauli matrices

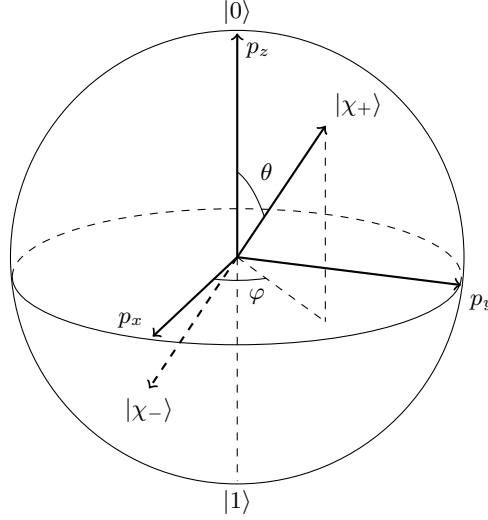


Figure 1. Unit Bloch Sphere. The computational basis is mapped to the north and south poles of the sphere. The orthogonal pure states $|\chi_+\rangle$ and $|\chi_-\rangle$ are antipodal and they can be described in terms of the polar angles (θ, φ) and the computational basis, as in Eqs. (1) and (5).

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2)$$

the computational basis states are the eigenstates of σ_3

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3)$$

Hence, the qubit in Eq. (1) can be rewritten as

$$|\chi_+\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}. \quad (4)$$

Points on the surface of the Bloch sphere connected by a diameter correspond to orthogonal pure states, and the antipodal state to $|\chi_+\rangle$ is given by

$$|\chi_-\rangle = -e^{-i\varphi} \sin\left(\frac{\theta}{2}\right) |0\rangle + \cos\left(\frac{\theta}{2}\right) |1\rangle, \quad (5)$$

or equivalently

$$|\chi_-\rangle = \begin{pmatrix} -e^{-i\varphi} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \quad (6)$$

The qubits $|\chi_+\rangle$ and $|\chi_-\rangle$ form an orthonormal set.

Let us now consider the free, massless Dirac equation

$$i\gamma^\mu \partial_\mu \Psi = 0, \quad (7)$$

where the gamma matrices $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ satisfy the Clifford algebra relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$, with $g^{\mu\nu}$ the metric tensor with signature $\text{diag}(1, -1, -1, -1)$. Using the Weyl representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad (8)$$

here denoted in 2×2 block form, with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, Eq. (7) decouples into the two Weyl equations[9–11]

$$i \frac{\partial}{\partial t} \psi_+ = -i \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \psi_+, \quad (9)$$

$$i \frac{\partial}{\partial t} \psi_- = i \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \psi_-. \quad (10)$$

Inserting the plane wave solutions

$$\psi_{\pm} = \chi_{\pm}(\mathbf{p}) \exp \{-i (Et - \mathbf{x} \cdot \mathbf{p})\}, \quad (11)$$

into Eqs. (9) and (10), respectively, we get

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \chi_{\pm}(\mathbf{p}) = \pm \chi_{\pm}(\mathbf{p}), \quad (12)$$

with $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ and $p^0 = E = |\mathbf{p}|$. Thus, the spinors $\chi_{\pm}(\mathbf{p})$ are helicity eigenstates, with the sign labeling the helicity. Representing the three-momentum in spherical polar coordinates

$$\hat{\mathbf{p}} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \quad (13)$$

the following solutions to Eq. (12) are obtained

$$\begin{aligned} \chi_+(\mathbf{p}) &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{pmatrix}, \\ \chi_-(\mathbf{p}) &= \begin{pmatrix} -e^{-i\varphi} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \end{aligned} \quad (14)$$

which are just Eqs. (4) and (6), hence the subscripts in the latter equations. The relation between massless Weyl spinors and orthogonal pure state qubits is now established: if the Bloch vector represents the physical three-momentum $\hat{\mathbf{p}}$, any two pair of antipodal pure state qubits are just helicity eigenstates, which in turn are momentum space solutions to the Weyl equations.

The spinors in Eq. (14) are not unique, since any other pair that differs by an overall phase is also a solution to Eq. (12). One such other pair of orthonormal helicity spinors, often found in the literature, is given by[21, 22]

$$\begin{aligned} \eta_+(\mathbf{p}) &= \begin{pmatrix} e^{-i\varphi/2} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi/2} \sin\left(\frac{\theta}{2}\right) \end{pmatrix} \\ \eta_-(\mathbf{p}) &= \begin{pmatrix} -e^{-i\varphi/2} \sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi/2} \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \end{aligned} \quad (15)$$

which correspond to the qubits[23, 24]

$$|\eta_+\rangle = e^{-i\varphi/2} \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi/2} \sin\left(\frac{\theta}{2}\right) |1\rangle, \quad (16)$$

$$|\eta_-\rangle = -e^{-i\varphi/2} \sin\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi/2} \cos\left(\frac{\theta}{2}\right) |1\rangle. \quad (17)$$

These qubits are related to $|\chi_{\pm}\rangle$ by a phase

$$|\chi_{\pm}\rangle = e^{\pm i\varphi/2} |\eta_{\pm}\rangle, \quad (18)$$

and hence we would not expect, in principle, any physical difference with respect to $|\chi_{\pm}\rangle$. However, they transform differently under two successive parity transformations[21, 25, 26]. To show it, let us implement twice the standard discrete parity transformation \mathcal{P}

$$\mathcal{P} : (\theta, \varphi) \rightarrow (\pi - \theta, \phi + \pi), \quad (19)$$

on the four qubit states. The results are

$$|\chi_{\pm}\rangle \xrightarrow{\mathcal{P}} \mp e^{\pm i\varphi} |\chi_{\mp}\rangle \xrightarrow{\mathcal{P}} |\chi_{\pm}\rangle, \quad (20)$$

$$|\eta_{\pm}\rangle \xrightarrow{\mathcal{P}} i |\eta_{\mp}\rangle \xrightarrow{\mathcal{P}} -|\eta_{\pm}\rangle. \quad (21)$$

Thus, $\mathcal{P}^2 = -1$ for qubits $|\eta_{\pm}\rangle$, while $\mathcal{P}^2 = 1$ for $|\chi_{\pm}\rangle$. [27] The density matrices are just helicity projection operators, and they do not distinguish between states of different \mathcal{P}^2

$$\begin{aligned} \rho_+ &= |\chi_+\rangle \langle \chi_+| = |\eta_+\rangle \langle \eta_+| = \frac{1}{2} (I_2 + \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}), \\ \rho_- &= |\chi_-\rangle \langle \chi_-| = |\eta_-\rangle \langle \eta_-| = \frac{1}{2} (I_2 - \boldsymbol{\sigma} \cdot \hat{\mathbf{p}}), \end{aligned} \quad (22)$$

where I_2 is the 2×2 identity matrix.

III. TWO LEVEL-SYSTEM UNITARY TRANSFORMATIONS

As previously stated, it is not possible to change the general qubit $|\chi_+\rangle$ ($|\eta_+\rangle$) into the orthogonal state $|\chi_-\rangle$ ($|\eta_-\rangle$) with a single unitary *constant* matrix, but this can be done with an anti-unitary transformation [2, 3, 28]. For Weyl spinors this is known as the Wigner transformation[15–17]

$$\begin{aligned} -i\sigma_2\chi_+^* &= +\chi_-, \\ -i\sigma_2\chi_-^* &= -\chi_+, \end{aligned} \quad (23)$$

which is anti-unitary because of complex conjugation. For qubits, the anti-unitary character of the Wigner transformation means that it cannot be realized as a quantum gate.

These results notwithstanding, it is well known that $SU(2)$ provides a double cover of $SO(3)$ because the latter is not simply connected. This means that there are closed loops in the $SO(3)$ sphere, the group manifold, which cannot be continuously reduced to a point, with the archetypical example being the closed loop which connects two antipodal points on the surface of the sphere, just as in the case of orthogonal pure state qubits. Given that $SO(3)$ is the three-dimensional rotation group, and that is isomorphic to the group of unitary transformations up to a phase, the so called projective representation group, one would expect that a unitary *continuous* transformation connecting the qubits can actually be obtained. The $SO(3)$ manifold is S^3 , a four dimensional unit sphere, but we can obtain a relation to the standard three-dimensional unit sphere S^2 (the Bloch sphere) by means of the quotient group $SO(3)/SO(2)$ [29, 30], then we have the following equivalence

$$SU(2)/U(1) \simeq SO(3)/SO(2) \simeq \mathbb{CP}^1 \simeq S^2, \quad (24)$$

where \mathbb{CP}^1 is the complex projective line, which is just the Hilbert space of a two-level quantum system. Equation (24) provides the group theoretical basis for the existence of unitary transformations connecting orthogonal pure state qubits, although a rigorous mathematical justification requires further analysis. Regardless, the transformations can be explicitly given. In this section I present a general realization of such transformations, and specific examples for qubits/Weyl spinors will be given in the following section.

Let us consider the following orthonormal states

$$\begin{aligned} |\Psi\rangle &= \alpha |0\rangle + \beta |1\rangle, \\ |\Psi_{\perp}\rangle &= -\beta^* |0\rangle + \alpha^* |1\rangle, \end{aligned} \quad (25)$$

where $|\alpha|^2 + |\beta|^2 = 1$, and $\{|0\rangle, |1\rangle\}$ is the basis of a two-level system. From these states, the following matrix is obtained

$$\Pi = \delta_1 |\Psi\rangle \langle \Psi_\perp| + \delta_2 |\Psi_\perp\rangle \langle \Psi|, \quad (26)$$

with δ_1 and δ_2 arbitrary phases. In the basis of Eq. (25) the explicit matrix form is

$$\Pi = \begin{pmatrix} \langle \Psi | \Pi | \Psi \rangle & \langle \Psi | \Pi | \Psi_\perp \rangle \\ \langle \Psi_\perp | \Pi | \Psi \rangle & \langle \Psi_\perp | \Pi | \Psi_\perp \rangle \end{pmatrix} = \begin{pmatrix} 0 & \delta_1 \\ \delta_2 & 0 \end{pmatrix}. \quad (27)$$

Hence,

$$\det \Pi = -\delta_1 \delta_2. \quad (28)$$

Taking the Hermitian conjugate of Π gives

$$\Pi^\dagger = \delta_2^* |\Psi\rangle \langle \Psi_\perp| + \delta_1^* |\Psi_\perp\rangle \langle \Psi|, \quad (29)$$

and so

$$\Pi \Pi^\dagger = \Pi^\dagger \Pi = \begin{pmatrix} |\delta_1|^2 & 0 \\ 0 & |\delta_2|^2 \end{pmatrix}. \quad (30)$$

The action of Π on the states in Eq. (25) is easily obtained

$$\begin{aligned} \Pi |\Psi\rangle &= \delta_2 |\Psi_\perp\rangle, \\ \Pi |\Psi_\perp\rangle &= \delta_1 |\Psi\rangle. \end{aligned} \quad (31)$$

If we now make the choices

$$\begin{aligned} \delta_1 \delta_2 &= -1, \\ |\delta_1|^2 &= |\delta_2|^2 = 1, \end{aligned} \quad (32)$$

then from Eqs. (28) and (30) we have that Π is a unitary matrix with unit determinant, and thus is a pure rotation belonging to $SU(2)$. Moreover, because of the antipodal character of the orthogonal states and the results in Eq. (31), we obtain the remarkable result that Π produces a point inversion through the origin of the Bloch sphere by means of a rotation. It must be emphasized that this result does not contradict the assertion at the beginning of this section, because Π is coordinate dependent by construction, so rather than having a single constant matrix we obtain a family of matrices, one for each pair of orthogonal pure states, which transforms the states into each other up to phases, and by imposing the conditions in Eq. (32) a subset of that family is made up of parity changing rotations.

IV. APPLICATIONS TO QUBITS

A. NOT/parity quantum gates

The results of the previous section can now be readily applied to the qubits $|\chi_\pm\rangle$ and $|\eta_\pm\rangle$. Let us first consider the matrices

$$\begin{aligned} P_1 &= e^{-i\varphi} |\chi_+\rangle \langle \chi_-| - e^{i\varphi} |\chi_-\rangle \langle \chi_+|, \\ P_2 &= i |\eta_+\rangle \langle \eta_-| + i |\eta_-\rangle \langle \eta_+|, \end{aligned} \quad (33)$$

which are unitary and of unit determinant as can be checked with the aid of Eq. (32). In matrix form they are given by

$$P_1 = \exp\left(i\frac{\pi}{2}\mathbf{a} \cdot \boldsymbol{\sigma}\right) = \begin{pmatrix} 0 & e^{-i\varphi} \\ -e^{i\varphi} & 0 \end{pmatrix}, \quad (34)$$

$$P_2 = \exp\left(i\frac{\pi}{2}\mathbf{b} \cdot \boldsymbol{\sigma}\right) = \begin{pmatrix} -i\sin\theta & ie^{-i\varphi}\cos\theta \\ ie^{i\varphi}\cos\theta & i\sin\theta \end{pmatrix}, \quad (35)$$

where the vectors

$$\begin{aligned} \mathbf{a} &= (-\sin\varphi, \cos\varphi, 0), \\ \mathbf{b} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \end{aligned} \quad (36)$$

define the respective rotation axis. Acting on the corresponding qubits they yield

$$\begin{aligned} P_1 |\chi_{\pm}\rangle &= \mp e^{\pm i\varphi} |\chi_{\mp}\rangle, \\ P_2 |\eta_{\pm}\rangle &= i |\eta_{\mp}\rangle, \end{aligned} \quad (37)$$

that is just the parity transformation in Eq. (19). Thus, they constitute a family of NOT/parity quantum gates implemented by rotations. The transformation phases can be adjusted by properly choosing the phases in Eq. (26), and another interesting choice is given by the matrices

$$\begin{aligned} P_3 &= -|\chi_+\rangle\langle\chi_-| + |\chi_-\rangle\langle\chi_+|, \\ P_4 &= -|\eta_+\rangle\langle\eta_-| + |\eta_-\rangle\langle\eta_+|, \end{aligned} \quad (38)$$

that also belong to $SU(2)$ and which realize the phase transformation in Eq. (23)

$$\begin{aligned} P_3 |\chi_{\pm}\rangle &= \pm |\chi_{\mp}\rangle, \\ P_4 |\eta_{\pm}\rangle &= \pm |\eta_{\mp}\rangle. \end{aligned} \quad (39)$$

As a final example, choosing the trivial phases $\delta_1 = \delta_2 = 1$ in Eq. (26) we obtain the matrices

$$\begin{aligned} \tilde{P}_1 &= |\chi_+\rangle\langle\chi_-| + |\chi_-\rangle\langle\chi_+|, \\ \tilde{P}_2 &= |\eta_+\rangle\langle\eta_-| + |\eta_-\rangle\langle\eta_+|, \end{aligned} \quad (40)$$

that are still unitary, but now $\det \tilde{P}_1 = \det \tilde{P}_2 = -1$, so they do not correspond to pure rotations. Their action on the qubits is as expected

$$\begin{aligned} \tilde{P}_1 |\chi_{\pm}\rangle &= |\chi_{\mp}\rangle, \\ \tilde{P}_2 |\eta_{\pm}\rangle &= |\eta_{\mp}\rangle. \end{aligned} \quad (41)$$

Besides being unitary, the matrices P_1 through P_4 are anti-Hermitian, while \tilde{P}_1 and \tilde{P}_2 are Hermitian, so we get the relations

$$\begin{aligned} P_i^\dagger &= P_i^{-1} = -P_i \quad \text{for } i = 1, \dots, 4, \\ \tilde{P}_i^\dagger &= \tilde{P}_i^{-1} = \tilde{P}_i \quad \text{for } i = 1, 2. \end{aligned} \quad (42)$$

It should also be noticed that although P_1 and P_2 effectively realize a parity transformation, they do not have the same effect as \mathcal{P}^2 for the states χ_{\pm} . In fact, all four matrices from P_1 to P_4 square to -1 , which confirm their rotational character, since it is a well known fact that spinors change sign under a full rotation that returns them to the starting point.

Because of Eq. (18), P_1 can also act on the states $|\eta_{\pm}\rangle$, yielding

P_1	$ \chi_+\rangle$	$ \chi_-\rangle$	$ \eta_+\rangle$	$ \eta_-\rangle$	$ 0\rangle$	$ 1\rangle$
	$-e^{i\varphi} \chi_-\rangle$	$e^{-i\varphi} \chi_+\rangle$	$- \eta_-\rangle$	$ \eta_+\rangle$	$-e^{i\varphi} 1\rangle$	$e^{-i\varphi} 0\rangle$

Table I. Action of the P_1 matrix on the two types of qubits/Weyl spinors and the computational basis.

$$P_1 |\eta_{\pm}\rangle = \mp |\eta_{\mp}\rangle. \quad (43)$$

As for the computational basis we have

$$\begin{aligned} P_1 |0\rangle &= -e^{i\varphi} |1\rangle, \\ P_1 |1\rangle &= e^{-i\varphi} |0\rangle. \end{aligned} \quad (44)$$

Collecting the results in Eqs. (37), (43), and (44), summarized in Table I, we see that P_1 constitutes a one parameter family of transformations with the property, not shared with any of the other transformations, of producing the orthogonal state up to a phase to any given pure state, including the computational basis, and in this sense it can be regarded as a universal quantum NOT gate.

B. Parity rotations

Denoting the SU(2) matrices P_1 through P_4 collectively as P , it is straightforward to verify that

$$P\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} P^\dagger = -\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}, \quad (45)$$

and by the well known relation between the groups SU(2) and SO(3)[13, 15], we expect the left-hand side of Eq. (45) to induce a rotation on the three-vector \mathbf{p} . Indeed, using the mapping[31]

$$R(P)_{ij} = \frac{1}{2} \text{Tr}(\sigma_i P \sigma_j P^\dagger), \quad (46)$$

the induced SO(3) rotation can be worked out for all the P matrices. E.g., for P_1 we obtain

$$R(P_1) = \begin{pmatrix} -\cos 2\varphi & -\sin 2\varphi & 0 \\ -\sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (47)$$

which is orthogonal and of unit determinant, and so it belongs to SO(3). Its action on \mathbf{p} given by Eq. (13), expressed as a column vector, yields

$$R(P_1) \mathbf{p} = -\mathbf{p}. \quad (48)$$

The same result holds for the rest of the P matrices induced rotations, so we can generally write

$$R(P) \mathbf{p} = -\mathbf{p}. \quad (49)$$

Thus, these matrices effectively provide a parity transformation by means of a rotation, a result which is most unexpected since the standard parity transformation in three-dimensional Euclidean space is provided by $\mathcal{P} = -I_3$, minus the three-dimensional identity matrix, and is a discrete transformation, not continuously connected to the identity, and hence a member of O(3) but not of SO(3). What, then, is the difference between these two types of parity transformations? It is clear from their construction that the P matrices are given in the same coordinate system that the spinors/qubits they act upon. This is also true for the induced $R(P)$ rotations acting on three-vectors. In fact, it can be verified that substituting the inverse mapping

$$\begin{aligned}\theta &= \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \\ \varphi &= \arctan\left(\frac{y}{x}\right)\end{aligned}\tag{50}$$

in the $R(P)$ matrices, and applying them to the column coordinate vector $\mathbf{x} = (x, y, z)$, results in $-\mathbf{x}$. This is in contrast with the standard $-I_3$ parity transformation, which produces $-\mathbf{x}$ *independently* of the coordinate system used to express \mathbf{x} . As for the \bar{P} matrices, they also satisfy Eq. (45), but they belong to $U(2)$ and cannot induce an $SO(3)$ transformation, nor can they be pure rotations.

If the Bloch vector is not the three-momentum, but instead a general vector $\hat{\mathbf{n}}$ with the same coordinates as in Eq. (13), all the results presented thus far still hold, with the proviso that in this case the qubits/spinors are no longer helicity eigenstates, but rather fixed-axis, non-relativistic spinors[22, 32], and therefore not solutions to the Weyl equations. This also shows that indeed the P matrices can be regarded as quantum negation gates, independently of the character of the Bloch vector.

C. Dynamical decoupling

Let us consider a qubit interacting with the environment (the bath), with the general interacting Hamiltonian

$$\mathcal{H}_{SB} = \boldsymbol{\sigma} \cdot \mathbf{B},\tag{51}$$

where $\mathbf{B} = (B_1, B_2, B_3)$ is the bath vector operator. The evolution of the qubit through the bath (free evolution) is then given by

$$U(t) = \exp(-i\mathcal{H}_{SB}t),\tag{52}$$

If \mathbf{B} is given in the same coordinates as \mathbf{p} in Eq. (13)

$$\mathbf{B} = |\mathbf{B}|(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\tag{53}$$

then it follows from Eq. (45) that

$$\{P, \mathcal{H}_{SB}\} = 0,\tag{54}$$

where the curly brackets indicate anti-commutation. Let us now consider the following cycle: free evolution for a time τ followed by an application of a P^\dagger transformation on the qubit, followed by free evolution for another interval of time τ , followed by an application of a P transformation on the qubit. Using the relation $A \exp(iB)A^\dagger = \exp(iABA^\dagger)$ and Eq. (54) the evolution through the cycle is

$$P \exp(-i\mathcal{H}_{SB}t) P^\dagger = I_2,\tag{55}$$

meaning the system is perfectly decoupled from the bath. The action of the P transformations is to be understood here as the application of pulses on the system. They can be realized as a magnetic field applied in the direction of the given rotation axis, e.g. the vector \mathbf{a} in Eq. (34), provided the magnetic field vector is properly normalized to preserve the unitarity of P . These pulses are localized because of the coordinate dependence of the P transformations, and can be made almost instantaneous by a sufficiently strong magnetic field.

V. WEYL EQUATIONS AND PARITY ROTATIONS

To simplify the notation, let the spinors, either $\chi_\pm(\mathbf{p})$ or $\eta_\pm(\mathbf{p})$, be generally denoted by $\xi_\pm(\mathbf{p})$. When acted upon by the P matrices, the phases in going from left-handed spinors to right-handed ones will be denoted by $\lambda(\mathbf{p})$, and those in going from right-handed spinors to left-handed ones by $\lambda'(\mathbf{p})$. Thus,

$$\begin{aligned} P\xi_- &= \lambda(\mathbf{p})\xi_+, \\ P\xi_+ &= \lambda'(\mathbf{p})\xi_-, \end{aligned} \quad (56)$$

and Eq. (12) reads

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \xi_+(\mathbf{p}) = \xi_+(\mathbf{p}), \quad (57)$$

for a positive energy solution to Eq. (9). The same equation is satisfied by the negative energy solution $\xi_+(\mathbf{p}) \exp \{i(Et + \mathbf{x} \cdot \mathbf{p})\}$ [11]. Similarly, solutions to Eq. (10) are given by

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \xi_-(\mathbf{p}) = -\xi_-(\mathbf{p}). \quad (58)$$

From Eq. (57) we have

$$P\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} P^\dagger P\xi_+(\mathbf{p}) = P\xi_+(\mathbf{p}), \quad (59)$$

and upon using Eqs. (56) and (45) we get back Eq. (58). In this manner we obtain a *unitary* relation between the right and left-handed Weyl equations. This is to be contrasted with Wigner's anti-unitary case, with $P\xi_\pm(\mathbf{p})$ replacing $\sigma_2 \xi_\pm^*(\mathbf{p})$, and Eq. (45) replacing $\sigma_2 (\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})^* \sigma_2 = -\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$.

In four dimensional space-time the P matrices do not induce a transformation on four-vectors. This is because the relation between $\text{SL}(2, C)$, the set of 2×2 complex matrices with unit determinant, and the restricted Lorentz group L_+^\uparrow [13, 14], consisting of rotations and Lorentz boosts, breaks down for the massless case in consideration. This relation is established through the identification of the four-momentum vector $p^\mu = (E, \mathbf{p})$ with the Hermitian matrix

$$\sigma \cdot p = \begin{pmatrix} E + |\mathbf{p}| \cos \theta & e^{-i\varphi} |\mathbf{p}| \sin \theta \\ e^{i\varphi} |\mathbf{p}| \sin \theta & E - |\mathbf{p}| \cos \theta \end{pmatrix}, \quad (60)$$

where $\sigma \cdot p \equiv \sigma^\mu p_\mu$, $\sigma^\mu = (I_2, \boldsymbol{\sigma})$, and \mathbf{p} is given by Eq. (13). Then the similarity transformation $A\sigma \cdot pA^\dagger$, with $A \in \text{SL}(2, C)$ corresponds to the transformation Λp with $\Lambda \in L_+^\uparrow$. But $\det \sigma \cdot p = \det A\sigma \cdot pA^\dagger = p^2 = 0$ in the massless case, and this would induce a non-invertible transformation on four-vectors, which cannot be a Lorentz transformation, whether it belongs to the restricted group or not.

Equation (60) also appears in the definition of twistors [12]. Changing the spinors normalization from unity to $\sqrt{2E} = \sqrt{2|\mathbf{p}|}$ we readily obtain the defining relation

$$\sigma \cdot p = 2|\mathbf{p}| \xi_+ \xi_+^\dagger, \quad (61)$$

for the twistors $\sqrt{2|\mathbf{p}|} \xi_+$ and $\sqrt{2|\mathbf{p}|} \xi_+^\dagger$, which are just positive helicity spinors/qubits and their Hermitian conjugates. On the other hand, transforming $\sigma \cdot p$ with the P matrices gives an alternative definition in terms of $\sqrt{2|\mathbf{p}|} \xi_-$ and $\sqrt{2|\mathbf{p}|} \xi_-^\dagger$

$$P\sigma \cdot pP^\dagger = \begin{pmatrix} E - |\mathbf{p}| \cos \theta & -e^{-i\varphi} |\mathbf{p}| \sin \theta \\ -e^{i\varphi} |\mathbf{p}| \sin \theta & E + |\mathbf{p}| \cos \theta \end{pmatrix} = 2|\mathbf{p}| \xi_- \xi_-^\dagger, \quad (62)$$

whose spatial part is just Eq. (45).

VI. CONCLUDING REMARKS

In this letter I have shown a new class of transformations with the remarkable property of effecting a space inversion while being pure rotations, and use them to realize an operation thought not to be possible before, namely, the fact that orthogonal pure state qubits can be unitarily transformed into each other, and the same for Weyl spinors of

different helicity. For qubits, the transformations provide a new kit of parity gates, and in one instance they act as universal NOT gates. They also serve the purpose of canceling the coupling with the environment in dynamical decoupling schemes. For Weyl spinors they provide a new symmetry transformation for the Weyl equations. I have also shown an equivalence between qubits and Weyl spinors which could foster interdisciplinary research.

The transformations here presented require a complete characterization within the theory of Lie groups, and by themselves constitute a relevant result in mathematical physics, but they could also find useful applications in other areas where this theory is relevant, such as condensed matter and high-energy physics. There is also the possibility to apply them in quantum optics, and to use them to improve other areas of quantum information, such as the fidelity of quantum anti-cloning.

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